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An alternative path to the Kruskal extension of the Schwarzschild metric

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Abstract. We show how it is relatively straightforward to deduce the Kruskal line element directly from the vacuum field equations, thereby avoiding the conventional approach which sets out from the analytic continuation of the Schwarzschild metric.

1. Introduction

Present discussions leading to the Kruskal (1960) and Szekeres (1960) maximal space–time for spherically symmetric vacuum fields invariably follow the sequence of historical development. Thus one obtains first the usual Schwarzschild exterior metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \, d\Omega^2, \tag{1}$$

discovers that $r = 2m$ is a coordinate singularity and then proceeds to eliminate it by an appropriate coordinate transformation $(t, r) \to (v, u)$ which exhibits the full geometric structure.

Although this sequence involving analytic continuation (see Misner *et al* 1973, Adler *et al* 1975) has proved to be the natural one historically, there appears to be an alternative way of proceeding which avoids the analytic continuation step in the sequence. This procedure, which we outline below, is to solve the field equations directly in terms of Kruskal type coordinates v, u and then invoke the uniqueness implied by the Birkhoff theorem to identify the resulting solution with the Schwarzschild metric (1). The consequent complete display of the topology of spherically symmetric vacuum space–time then proceeds as in the conventional sequence of discussion.

Our work below, outlining the alternative step to coordinate transformations, turns out to be relatively simple mathematically and in our opinion affords some new insights into this most fundamental of problems in general relativity. We feel it should represent a useful approach complementary to the conventional one.

2. Direct calculation of the Kruskal metric

We define Kruskal type coordinates v, u by insisting that the metric in these coordinates take the particular non-static spherically symmetric form

$$ds^2 = A(dv^2 - du^2) - B \, d\Omega^2. \tag{2}$$

Here A and B are to be positive non-vanishing functions of v and u, with B clearly coinciding with the square of the radial marker r in the Schwarzschild metric (1). Radial light rays ($\mathrm{d}s^2 = 0$, $\mathrm{d}\theta = \mathrm{d}\phi = 0$) are here simply straight lines $v = \pm u$ and consequently the possible world lines of material test particles and photons can be immediately envisaged in the (v, u) space–time diagram.

We substitute the metric (2) into the vacuum field equation $R_{\mu\nu} = 0$. There are only four independent non-vanishing Ricci tensor components R_{00}, R_{11}, R_{22} $(= \sin^2\theta R_{33})$, R_{01}. By convenient linear combinations we write the four field equations as

$$R_{00}+R_{11} = -\frac{\ddot{B}+\overset{**}{B}}{B}+\frac{\dot{B}^2+\overset{*}{B}{}^2}{2B^2}+\frac{\dot{A}\dot{B}+\overset{*}{A}\overset{*}{B}}{AB}=0 \tag{3}$$

$$R_{00}-R_{11} = -\frac{\ddot{A}-\overset{**}{A}}{A}-\frac{\ddot{B}-\overset{**}{B}}{B}+\frac{\dot{B}^2-\overset{*}{B}{}^2}{2B^2}+\frac{\dot{A}^2-\overset{*}{A}{}^2}{A^2}=0 \tag{4}$$

$$R_{22} = \frac{\ddot{B}-\overset{**}{B}}{2A}+1=0 \tag{5}$$

$$R_{01} = -\frac{\dot{\overset{*}{B}}}{B}+\frac{\dot{B}\overset{*}{B}}{2B^2}+\frac{\dot{A}\overset{*}{B}+\overset{*}{A}\dot{B}}{2AB}=0 \tag{6}$$

with 'dots' and 'stars' denoting partial derivatives with respect to v and u, respectively. As the next step, we impose a more stringent coordinate condition than that already implicitly adopted via the assumed metric form (2), by requiring the unknown functions A and B to involve the variables v, u only in the general and as yet unspecified coordinate combination

$$\xi = \xi(v, u),$$

that is

$$A(v, u) = A(\xi) \qquad B(v, u) = B(\xi).$$

An examination of the resulting field equations then suggests that a considerable simplification may result from a judicious choice of the function ξ. Indeed, a little experimentation shows that the simple antisymmetric choice

$$\xi = v^2 - u^2 \tag{7}$$

converts the field equations to a non-trivial consistent set in the *single* variable ξ. This simplification, in fact, reduces the four field equations to only *three* in number (equations (3) and (6) become the same equation). Apart from non-vanishing factors which we omit, they are

$$\frac{B''}{B}-\frac{B'^2}{2B^2}-\frac{A'B'}{AB}=0 \tag{8}$$

$$2(B'+\xi B'')+A=0 \tag{9}$$

$$\frac{A'+\xi A''}{A}+\frac{B'+\xi B''}{B}-\frac{\xi B'^2}{2B^2}-\frac{\xi A'^2}{A^2}=0. \tag{10}$$

Here 'primes' denote $\mathrm{d}/\mathrm{d}\xi$.

The above are three ordinary differential equations coupling two unknown functions A and B of the single variable ξ. We can readily obtain an integral of equation

(8), namely

$$(B^{1/2})' = KA \tag{11}$$

where K is an arbitrary integration constant. If we use this in turn to eliminate A in equation (9), we can integrate the resulting equation completely to obtain the relation

$$\xi = H(B^{1/2} + L)^{4KL}\, \mathrm{e}^{-4KB^{1/2}}. \tag{12}$$

Here L and H are two further integration constants. Writing for convenience

$$\rho \equiv B^{1/2} \tag{13}$$

(we reserve the symbol $r \equiv |\rho|$ for the Schwarzschild radial marker) we then have the two equations (11) and (12), i.e.

$$\rho' = KA \tag{11$'$}$$

$$\xi = H(\rho + L)^{4KL}\, \mathrm{e}^{-4K\rho}, \tag{12$'$}$$

together with the third field equation (10), which up to this point has not yet been used.

By differentiating (12′) with respect to ρ to get

$$\frac{\mathrm{d}\xi}{\mathrm{d}\rho} = -4KH\rho(\rho + L)^{4KL-1}\, \mathrm{e}^{-4K\rho} \tag{14}$$

and substituting into (11′), we immediately obtain a solution for the function A as

$$A = \left(-\frac{1}{4K^2H}\right)\frac{(\rho + L)^{1-4KL}}{\rho}\, \mathrm{e}^{4K\rho}. \tag{15}$$

This solution in turn satisfies the third field equation (10) as can be seen by direct substitution. Thus in (12′) and (15) we have the complete and consistent solutions of the Einstein field equations with three arbitrary integration constants H, K and L.

The metric functions A and B are, however, only implicitly defined in terms of the variable $\xi \equiv v^2 - u^2$. In fact, we can only invert equation (12′) so as to obtain an explicit single-valued expression for $\rho (\equiv B^{1/2})$ in terms of ξ, over *monotonic* sections of the $\xi - \rho$ curve (12′), and hence only over such monotonic sections can we expect the metric elements A and B to remain single-valued functions of ξ. With this monotonicity in mind, we look at the derivative $\mathrm{d}\xi/\mathrm{d}\rho$ in (14) and are strongly motivated to dispose of one of the arbitrary integration constants L, by setting

$$4KL = 1,$$

since this leaves the resulting function ξ only *one* turning point, namely at $\rho = 0$ (maximum or minimum depending on the remaining constants H and K). At the same time possible complex values for ξ resulting from $(\rho + L) < 0$ in (12′), are avoided; ξ is a *real* function of ρ everywhere. The regions $\rho > 0$ and $\rho < 0$ are thus clearly monotonic with the corresponding Schwarzschild radial marker $r \equiv |\rho|$ ranging from 0 to ∞ in both regions. Thus we are reduced to the two simpler solutions

$$\xi = H\left(\rho + \frac{1}{4K}\right)\mathrm{e}^{-4K\rho} \tag{16}$$

$$A = -\frac{1}{4K^2H}\frac{\mathrm{e}^{4K\rho}}{\rho} \tag{17}$$

which represent necessarily single-valued solutions for B and A as functions of ξ over the separate regions $\rho>0$ and $\rho<0$.

It remains now to fix one of the two constants H, K by normalisation and to examine distinct possibilities. Thus if we set

$$H=4K$$

and write, for purposes of comparison with the standard Kruskal form,

$$4K\equiv-1/2m$$

we get

$$\xi\equiv v^2-u^2=(1-\rho/2m)\,\mathrm{e}^{\rho/2m}$$

$$A=\frac{32m^3}{\rho}\,\mathrm{e}^{-\rho/2m}.$$

Two possibilities now arise from setting $m>0, \rho>0$ and $m>0, \rho<0$ ($m<0, \rho\lessgtr 0$ does not give anything essentially different).

(i) For $m>0$, $\rho>0$ we get, using the Schwarzschild coordinate $r\equiv|\rho|$, the well-known Kruskal relation

$$\xi\equiv v^2-u^2=(1-r/2m)\,\mathrm{e}^{r/2m}$$

$$A=\frac{32m^3}{r}\,\mathrm{e}^{-r/2m}. \tag{18}$$

Here the range of ξ is $(-\infty, 1)$ with $\xi\equiv v^2-u^2=1$ representing the singular hyperbolae beyond which test particles and light cannot progress.

(ii) For $m>0$, $\rho<0$ we have

$$\xi\equiv v^2-u^2=(1+r/2m)\,\mathrm{e}^{-r/2m}$$

$$A=-\frac{32m^3}{r}\,\mathrm{e}^{r/2m}$$

(which is just the solution (18) with $m\to-m$).

Here the metric component A is negative, indicating that u is now the time-like and v the space-like coordinate. Thus if we write

$$v'\equiv u \qquad u'\equiv v \qquad A'=-A$$

we have

$$\xi'\equiv v'^2-u'^2=-(1+r/2m)\,\mathrm{e}^{-r/2m}$$

$$A'=\frac{32m^3}{r}\,\mathrm{e}^{r/2m}. \tag{19}$$

The range of ξ' is here $(-1, 0)$.

The two solutions (18) and (19) must, by Birkhoff's theorem, correspond to the Schwarzschild solution (1) with $m>0$ and $m<0$. To decide which corresponds to which (and here we cannot arbitrate on this point *a priori* since we do not have a coordinate transformation) it is necessary only to examine radial freely falling test particles to show that the metric (18) corresponds to the metric (1) with $m>0$, and as a consequence (or independently) to deduce that (19) corresponds to the unphysical (gravitationally repulsive) case of metric (1) with $m<0$. This second case (19), quite apart from its unphysical Vmetric, is of no interest for the further reason that the corresponding Schwarzschild metric ($m<0$) has no coordinate singularity anyway and so gives a complete description of this unphysical space–time already.

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